Recitation 9

October 22, 2015

Review

Eigenvalues and eigenvectors: the following are equivalent

- λ is an eigenvector of a matrix A;
- $Av = \lambda v$ for some **non-zero** vector v;
- $(A - \lambda I)v = 0$ for some **non-zero** vector v;
- $(A - \lambda I)$ has non-zero null space;
- $(A - \lambda I)$ is not invertible;
- λ is a solution of the characteristic equation $\det(A - \lambda I) = 0$.

Algorithm for diagonalizing a matrix: suppose you need to diagonalize a matrix A (assume it can be diagonalized)

1. **Find eigenvalues.** To do that, solve the characteristic equation $\det(A - \lambda I) = 0$.
2. **Put eigenvalues into a matrix**. Namely, you write a **diagonal** matrix D with eigenvalues along the diagonal.
3. **Find eigenvectors.** For each λ that you have found, find a basis of eigenvectors, i.e. find a basis for $Nul(A - \lambda I)$. In other words, solve the homogeneous system $(A - \lambda I)v = 0$ and pick a basis for the space of solutions.
4. **Put the eigenvectors you have found in to a matrix.** Denote this matrix by P. **Note: the order of the vectors matters.** Eigenvalues λ and corresponding vectors in P should be in the same order. If you have several independent eigenvectors **for the same eigenvalue**, then the order among them doesn't matter. **Find the matrix** P^{-1}. Then $A = PDP^{-1}$. **Be happy.**

Algorithm for finding a matrix of linear transformation in two bases. Namely, suppose you have a transformation $T\colon V \to W$ from n-dimensional space to m-dimensional one. Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be a basis in V and $\mathcal{C} = \{w_1, \ldots, w_m\}$ be a basis in W. You need to find the matrix $M = [T]_{\mathcal{B},\mathcal{C}}$.

1. Find $T(v_1), \ldots, T(v_n)$.
2. Find the coordinates $[T(v_1)]_{\mathcal{C}}, \ldots, [T(v_n)]_{\mathcal{C}}$ of $T(v_1), \ldots, T(v_n)$ in the basis $\mathcal{C}$.
3. Put that into matrix: $M = [[T(v_1)]_{\mathcal{C}}, \ldots, [T(v_n)]_{\mathcal{C}}]$.

Algorithm to find matrix of $A\colon \mathbb{R}^n \to \mathbb{R}^m$ **in some bases.** The same thing as above really. If $\mathcal{B} = \{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, put these vectors into a matrix $P = [v_1 \ldots v_n]$. If $\mathcal{C} = \{w_1, \ldots, w_m\}$ is a basis of $\mathbb{R}^m$, put these vectors into a matrix $Q = [w_1 \ldots w_m]$. Then the matrix of A in non-standard bases $\mathcal{B}, \mathcal{C}$ is $M = Q^{-1}AP$.
If $n = m$ **and** $\mathcal{B} = \mathcal{C}$**:** the same stuff really. It's just now $P = Q$, so $M = P^{-1}AP$.
Notice: diagonalization is a particular case of that. If $P = [v_1 \ldots v_n]$ is a basis of **eigenvectors** of A, then $D = P^{-1}AP$ is the diagonalization of A corresponding to $\{v_1, \ldots, v_n\}$.

Problems

Problem 1. Find determinant of the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 4 & 1 \\ 4 & -5 & 2 \end{bmatrix}$

Problem 2. Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$.

Problem 3. Diagonalize the matrix $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$ knowing that the eigenvalues are $\lambda_1 = 1, \lambda_2 = 5$.

Problem 4. Can the matrix $B = \begin{bmatrix} 1 & 3 \\ -3 & 7 \end{bmatrix}$ be diagonalized?

Problem 5. Let $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Prove that these vectors form a basis of $\mathbb{R}^2$.
Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation $v \mapsto Av$ where $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. Find the matrix of T relative to the basis $\{v_1, v_2\}$.

Problem 6. Let $T\colon \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation sending a polynomial $p(t)$ to the polynomial $t \cdot p(t)'$.

1. Find the image of $p(t) = -t^2 + 2t + 1$.
2. Prove that this is a linear transformation.
3. Find the matrix of T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.
4. What is the null space of T? What is the rank of T?

Problem 7. Let a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation given in the standard bases by the matrix $A = \begin{bmatrix} 3 & 2 \\ 0 & -1 \\ 1 & 2 \end{bmatrix}$. Fix a non-standard basis in $\mathbb{R}^2$ given by $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and a non-standard basis in $\mathbb{R}^3$ given by $u_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ and $u_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$. Find the matrix of the transformation T relative to the bases $\{v_1, v_2\}$ and $\{u_1, u_2, u_3\}$.

Problem 8. Show that if an $n \times n$ matrix A has n linearly independent eigenvectors, then so does A^T. (Hint: use the Diagonalization theorem.)

Problem 9. Prove that if matrices A and B are similar, then they have equal characteristic polynomials. Will similar matrices have *the same* eigenvectors? What can you say about their eigenvectors?